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SOME INTEGRABLE EXTENSIONS OF JACOBI'S PROBLEM OF GEODESICS ON AN ELLIPSOID†

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The problem of a point moving on the surface of an n-dimensional ellipsoid in a conservative field of force is considered. It is shown that if the potential energy terms are inversely proportional to the squares of the distances to the $(n-1)$-dimensional planes of symmetry of the ellipsoid, the problem can be explicitly integrated by using separation of variables in elliptic Jacobi coordinates. It has n independent commuting integrals that are quadratic functions of the momenta. If $n = 2$, an additional integral can be found explicitly by using redundant coordinates. In the limit, when the least semi-axis approaches zero, one obtains a new integrable billiards problem inside the ellipse. Extensions of these results to a space of constant non-zero curvature are discussed.

1. THE MAIN RESULT

Jacobi [1] introduced elliptic coordinates in multidimensional Euclidean space and used them to solve a variety of non-trivial problems in dynamics. Among these is the celebrated problem of motion on the surface of an n-dimensional ellipsoid. The trajectories of a material point moving by inertia are geodesics. In addition, Jacobi used separation of variables to solve a more general problem in which the point is moving under the action of an elastic force whose centre is the centre of the ellipsoid.

Let $\mathbf{R}^{n+1}$ be Euclidean space with Cartesian coordinates $x_0, x_1, \ldots, x_n$. Consider an n-dimensional ellipsoid in $\mathbf{R}^{n+1}$

$$\sum_{s=0}^{n} \frac{x_s^2}{a_s^2} = 1 \tag{1.1}$$

where $a_0 < a_1 < \ldots < a_n$.

Theorem 1. The problem of motion on the ellipsoid (1.1) under the action of conservative forces with potential energy

$$V = \frac{k}{2}\sum_{s=0}^{n} x_s^2 + \sum_{\nu=0}^{n} \frac{\alpha_\nu}{x_\nu^2}; \quad k, \alpha_\nu = \text{const} \tag{1.2}$$

is completely integrable.

If $\alpha_0 = \ldots = \alpha_n = 0$, we obtain Jacobi's classical result [1]: the potential (1.2) generates a field of elastic forces whose centre is at the origin.

The potential (1.2) has a noteworthy property (cf. [2]). Consider the motion of a particle in $\mathbf{R}^{n+1}$, unconstrained by (1.1), under a conservative force with components $-\partial V/\partial x_\nu$. This problem can be readily solved by separation of Cartesian coordinates. It turns out that all bounded trajectories are closed. In actual fact, this is not quite correct: trajectories that reach the coordinate planes, where the function (1.1) has singularities, in a finite time must be excluded.

Theorem 1 also holds for numbers $a_0, \ldots, a_n$ of different signs. In that case Eq. (1.1) defines a hyperboloid in $\mathbf{R}^{n+1}$. In addition, the complete integrability property is preserved for ellipsoids of revolution, when some of the numbers $a_0, \ldots, a_n$ may be equal. The most interesting special case is that of $a_0 = \ldots = a_n = a$, when the ellipsoid (1.1) is a sphere $\mathbf{S}^n$ of radius $a^{1/2}$. Since the first term in (1.2) is constant on $\mathbf{S}^n$, it does not affect the dynamics of the particle.

The fact that the problem of motion on a two-dimensional sphere in a field of force with potential of the form (1.2) is integrable was pointed out in [3]. It was also shown that for potentials

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$$\alpha_\nu / x_\nu^2; \quad \nu = 0, 1, \ldots, n \tag{1.3}$$

almost all the orbits in S^n are closed. The functions (1.3) are analogues of the potential of an elastic spring in a space of constant positive curvature (one of the ends of the spring is attached at a point with coordinates $x_s = 0$ $(s \neq \nu), x_\nu = \pm 1$). As observed by Yu. N. Federov, the problem of a point moving on an n-dimensional sphere with potential energy (1.2) has $n(n+1)/2$ quadratic integrals

$$I_{ij} = (\dot{x}_i x_j - x_i \dot{x}_j)^2 + 2\alpha_i x_j^2 / x_i^2 + 2\alpha_j x_i^2 / x_j^2, \quad i, j = 0, 1, \ldots, n$$

Since $2n-1$ of these are independent, orbits with almost any initial data are closed.

2. SEPARATION OF VARIABLES

The ellipsoid (1.1) may be included in a family of confocal quadrics in $\mathbf{R}^{n+1}$

$$\sum_{s=0}^{n} \frac{x_s^2}{a_s - \lambda} = 1$$

As an algebraic equation in λ, this expression has exactly $n+1$ real roots

$$\lambda_0 < \lambda_1 < \ldots < \lambda_n \tag{2.1}$$

where $a_{s-1} < \lambda_s < a_s$ $(s \geqslant 1)$, $\lambda_0 < a_1$. The numbers (2.1) are elliptic coordinates in $\mathbf{R}^{n+1}$, related to Cartesian coordinates by the following formulae

$$x_\nu^2 = \prod_{s=0}^{n} (a_\nu - \lambda_s) / \prod_{s \neq \nu} (a_\nu - a_s) \tag{2.2}$$

Fix the value of the variable λ_0, say $\lambda_0 = 0$. We then obtain the ellipsoid (1.1). The other coordinates $\lambda_1, \ldots, \lambda_n$ will be Lagrange coordinates for the motion of a particle of unit mass on the surface (1.1). Let $\mu_1, \ldots, \mu_n$ be the conjugate momenta. The kinetic energy is [1]

$$T = 2\sum_{s=1}^{n} \frac{\mu_s^2}{M_s(\lambda)}, \quad M_s = -\lambda_s \prod_{\nu \neq s} \frac{\lambda_s - \lambda_\nu}{A(\lambda_s)}$$
$$A(z) = (z - a_0)(z - a_1) \ldots (z - a_n) \tag{2.3}$$

To simplify the notation, let us omit the first term in (1.2). It is easily taken into consideration by using Jacobi's formulae for separation of variables [1]. Using (2.2), we express the potential energy (1.2) in terms of the elliptic coordinates

$$V = \sum_{s=1}^{n} \frac{\beta_s}{(a_s - \lambda_1) \ldots (a_s - \lambda_n)} \tag{2.4}$$

where $\beta_0, \ldots, \beta_n$ are new constants. We will use the identity

$$\frac{1}{(a_s - \lambda_1) \ldots (a_s - \lambda_n)} = \sum_{j=1}^{n} \frac{\gamma_j}{a_s - \lambda_j}, \quad \gamma_j = \left(\prod_{\nu \neq j} (\lambda_j - \lambda_\nu) \right)^{-1} \tag{2.5}$$

easily established by using the Residue theorem. Using (2.4) and (2.5), we finally obtain

$$V = \sum_{s=1}^{n} \sum_{j=0}^{n} \frac{\beta_j \gamma_s}{a_j - \lambda_s} \tag{2.6}$$

Thus, by (2.3) and (2.6)

$$\sum_{s=1}^{n} \left[-2\mu_s^2 \frac{A(\lambda_s)}{\lambda_s} + \sum_{j=0}^{n} \frac{\beta_j}{a_j - \lambda_s} \right] \gamma_s = \sum_{s=1}^{n} [F_0 + F_1 \lambda_s + \ldots + F_{n-1} \lambda_s^{n-1}] \gamma_s \tag{2.7}$$

where $F_{n-1} = T + V$ is the total energy. According to Jacobi [1], the expression on the right is exactly F_{n-1}. Applying the general principle of separation of variables, let us equate the bracketed expressions in (2.7) for $s = 1, \ldots, n$. Since $\lambda_k \neq \lambda_l$ for $k \neq l$, it follows that the resulting linear system of equations will enable us to find $F_0, F_1, \ldots, F_{n-1}$ as quadratic functions of the momenta. Moreover, naturally, F_{n-1} is the total energy. The other functions $F_0, \ldots, F_{n-2}$ are commuting integrals of the problem.

Thus, the problem of motion on a n-dimensional ellipsoid with potential (1.2) has n independent integrals in involution, $F_0, \ldots, F_{n-1}$. By Liouville's theorem (see, for example, [2]), it is completely integrable. This proves the theorem.

Using the above linear system, one can write down differential equations for the elliptic coordinates

$$\begin{aligned} \lambda_s^{\cdot} &= \partial H / \partial \mu_s = \pm 4\gamma_s \sqrt{\Phi(\lambda_s)/(2\lambda_s)} \\ \Phi(z) &= A(z)[-F_0 - \ldots - F_{n-1}z^{n-1} + \sum \beta_j / (a_j - z)] \end{aligned} \tag{2.8}$$

Using the formula for A, we conclude that $\Phi(z)$ is a polynomial in z, of degree $2n$. System (2.8) has the form of the Abel–Kowalewski equations.

The complete integral of the Hamilton–Jacobi equation for our problem has the form

$$W = -F_{n-1}t + \frac{1}{2}\int \sqrt{\frac{\Phi(\lambda_s)}{\lambda_s}}\, d\lambda_s$$

The part of n arbitrary parameters $c_1, \ldots, c_n$ is played by the n independent integrals $F_0, \ldots, F_{n-1}$. The general solution of Eqs (2.8) is derived from the Jacobi relations

$$\partial W / \partial c_i = b_i, \quad b = \text{const}; \quad i = 1, \ldots, n$$

3. THE CASE OF TWO DEGREES OF FREEDOM

When $n = 2$ there is an additional quadratic integral that can be determined explicitly. Let x, y, z be Cartesian coordinates in $\mathbf{R}^3$ and

$$x^2/a + y^2/b + z^2/c = 1 \tag{3.1}$$

the equations of the ellipsoid. As shown by Joachimsthal (see, for example, [4]), the function

$$I = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(\frac{x^{\cdot 2}}{a} + \frac{y^{\cdot 2}}{b} + \frac{z^{\cdot 2}}{c}\right) \tag{3.2}$$

is an integral of motion by inertia.

Consider the motion of a particle of unit mass acted upon by a force with potential energy of the form (1.2)

$$V = \frac{k}{2}(x^2 + y^2 + z^2) + \frac{\alpha}{2x^2} + \frac{\beta}{2y^2} + \frac{\gamma}{2z^2} \tag{3.3}$$

We shall seek an integral of the equations of motion

$$x^{\cdot\cdot} = \lambda x / a - V_x, \quad y^{\cdot\cdot} = \lambda y / b - V_y, \quad z^{\cdot\cdot} = \lambda z / c - V_z \tag{3.4}$$

as a sum $F = I + f$, where f is an as yet unknown function of x, y, z.

Equations (3.1) and (3.4) yield the Lagrange multiplier λ as a function of the state of the particle

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\lambda = \frac{x}{a}V_x + \frac{y}{b}V_y + \frac{z}{c}V_z - \frac{x^{\cdot 2}}{a} - \frac{y^{\cdot 2}}{b} - \frac{z^{\cdot 2}}{c}$$

Since the function (3.2) is an integral of the equations of motion by inertia, it follows that all the terms in F that are cubic in the velocities $x^{\cdot}, y^{\cdot}, z^{\cdot}$ vanish together. The equation $F = 0$ takes the following explicit form

$$2\left(\frac{x}{a}V_x+\frac{y}{b}V_y+\frac{z}{c}V_z\right)=\left(\frac{xx^{\cdot}}{a^2}+\frac{yy^{\cdot}}{b^2}+\frac{zz^{\cdot}}{c^2}\right)-$$
$$-2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\left(\frac{x^{\cdot}}{a}V_x+\frac{y^{\cdot}}{b}V_y+\frac{z^{\cdot}}{c}V_z\right)+f_x x^{\cdot}+f_y y^{\cdot}+f_z z^{\cdot}=0 \tag{3.5}$$

If we equate the coefficients of $x^{\cdot}, y^{\cdot}, z^{\cdot}$ to zero, we obtain a system of three partial differential equations

$$2\left(\frac{x}{a}V_x+\frac{y}{b}V_y+\frac{z}{c}V_z\right)\frac{x}{a^2}-2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\frac{V_x}{a}=-f_x,\ \ldots \tag{3.6}$$

This system is readily solved for the singular part of the potential (3.3)

$$f=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\left(\frac{\alpha}{ax^2}+\frac{\beta}{by^2}+\frac{\gamma}{cz^2}\right) \tag{3.7}$$

However, if we substitute the function $V = k(x^2 + y^2 + z^2)/2$ into (3.6), we obtain an incompatible system of equations. In fact, however, there is no contradiction. The point is that, for an elastic potential, the second term in (3.5) vanishes owing to the identity

$$xx^{\cdot}/a+yy^{\cdot}/b+zz^{\cdot}/c=0$$

Hence we must omit the second terms in (3.6). Then, using (3.1), we find the solution

$$f=-k(x^2/a^2+y^2/b^2+z^2/c^2) \tag{3.8}$$

Summing (3.2), (3.7) and (3.8), we obtain the final result

$$F=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\left(\frac{x^{\cdot 2}}{a^2}+\frac{y^{\cdot 2}}{b^2}+\frac{z^{\cdot 2}}{c^2}-k+\frac{\alpha}{ax^2}+\frac{\beta}{by^2}+\frac{\gamma}{cz^2}\right)$$

4. INTEGRABLE BILLIARDS

Assuming that $c < b \leqslant a$, let the minor semi-axis c in Eqs (3.1) tend to zero. It is natural to expect the limiting problem to be that of a point moving inside the ellipse

$$x^2/a+y^2/b=1 \tag{4.1}$$

which rebounds elastically from its curve.

This passage to the limit was first studied, assuming no external forces, by Birkhoff [5] (see also [6]). The characteristic property of Birkhoff's billiards is as follows: the straight-line segments of all trajectories (or of their continuations) touch the same conic, that is confocal with the ellipse (4.1).

As shown in [7], addition of an attractive or repulsive elastic force with its centre at the origin will also produce an integrable billiard problem. This problem was analysed qualitatively (including the construction of bifurcation diagrams) in [8].

It has been shown [9] that application of a Bolin transformation to this problem produces a similar problem with a particle moving under a gravitational force directed toward the focus of the ellipse (4.1). In particular, gravitational elliptic billiards is also an integrable dynamical system.

Let us now let the semi-minor axis of the ellipsoid (3.1) approach zero in the more general problem of a particle moving in a field with potential (3.7). Clearly, as $c \to 0$ the z coordinate also tends to zero. In order to avoid the singularity at $z = 0$, we set $\gamma = 0$ in (3.7).

It can be shown that the limit

$$\lim_{c\to 0} cF1_{x,y} = \frac{x^{\cdot 2}}{a} + \frac{y^{\cdot 2}}{b} - \frac{(x^{\cdot}y - xy^{\cdot})^2}{ab} + \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(-k + \frac{\alpha}{ax^2} + \frac{\beta}{by^2}\right) \tag{4.2}$$

exists. The expression on the right of this quality is an integral of the billiard problem inside the ellipse (4.1), that is quadratic in the velocities. Consequently, the limiting problem is integrable. The trajectories of a particle inside the ellipse are made up of arcs of conic sections. This result can be generalized.

Theorem 2. Elastic billiards inside the ellipse (4.1) with potential

$$V = \frac{k}{2}(x^2 + y^2) + \frac{\alpha}{2x^2} + \frac{\beta}{2y^2} + \frac{\gamma_1}{r_1} + \frac{\gamma_2}{r_2} \tag{4.3}$$

were $k, \alpha, \beta, \gamma_1, \gamma_2$ are constants, and r_1 and r_2 are the distances from the foci of the ellipse to the particle, is an integrable dynamical system.

The proof employs the method of Section 3. Billiards in an ellipse with elastic rebounds is a system with two degrees of freedom. It admits of an energy integral $(x^{\cdot 2} + y^{\cdot 2})/2 + V$. For an additional integral we let (cf. (4.2))

$$F = x^{\cdot 2}/a + y^{\cdot 2}/b - (x^{\cdot}y - xy^{\cdot})^2/(ab) + f(x,y)$$

The component of this function that is quadratic in the velocities is an integral of Birkhoff billiards. Consequently, it remains unchanged at the instant of elastic collision (the velocity remains unchanged and the angle of incidence equals the angle of reflection). Therefore f should be sought subject to the condition that F is constant on the phase trajectories of the "free" system

$$x^{\cdot\cdot} = -V_x, \quad y^{\cdot\cdot} = -V_y$$

Equating the coefficients of $x^{\cdot}$ and $y^{\cdot}$ in the equation $F^{\cdot} = 0$ to zero, we obtain a system of two partial differential equations

$$\frac{2V_x}{a} - \frac{2y}{ab}(yV_x - xV_y) = f_x, \quad \frac{2V_y}{b} + \frac{2x}{ab}(yV_x - xV_y) = f_y$$

In view of the equality $f_{xy} = f_{yx}$, we obtain the desired second-order partial differential equation for the potential

$$(a-b)V_{xy} + 3(yV_x - xV_y) + (y^2 - x^2)V_{xy} + xy(V_{xx} - V_{yy}) = 0 \tag{4.4}$$

Corresponding to each solution of this equation we have an integrable billiards problem in the ellipse (4.1). Clearly, $a - b$ is the square of the distance from the focus to the centre of the ellipse. To complete the proof, it remains to verify that each term in (4.3) satisfies Eq. (4.4).

5. SOME EXTENSIONS

Let $\mathbf{R}^{n+2}$ be Euclidean space with Cartesian coordinates $x_1, \ldots, x_{n+2}$, and let

$$\mathbf{S}^{n+1} = \left\{x: \sum_{\nu=1}^{n+2} x_\nu^2 = 1\right\} \tag{5.1}$$

be an $(n+1)$-dimensional sphere, whose metric has constant positive curvature. We also consider in $\mathbf{R}^{n+2}$ an $(n+1)$-dimensional cone with apex the origin, defined by the equation

$$\sum_{\nu=1}^{n+2} \frac{x_\nu^2}{a_\nu - z} = 0 \tag{5.2}$$

where $a_1 < a_2 < \ldots < a_{n+2}$, $z \neq a_\nu$ and $a_1 < z < a_{n+2}$. The cone (5.2) cuts the sphere (5.1) in an n-dimensional surface E^n which is a natural analogue of an ellipsoid in a space of positive curvature.

As in the case of a flat space, the problem of geodesics on E^n is a completely integrable Hamiltonian system. This was essentially known to Jacobi [1]. Geometrical and analytical aspects of the problem of geodesics on E^n were discussed in [10]. This extension of Jacobi's problem is solved by separation of variables, using spheroconical coordinates. The latter are defined as the roots $z_1, \ldots, z_{n+1}$ of Eq. (5.2), which separates the numbers $a_1, \ldots, a_{n+2}$. The variables z_k are Lagrange coordinates on the sphere S^{n+1}. To express the Cartesian coordinates x in terms of z, one uses the equation of the sphere (5.1).

As already pointed out, the problem of inertial motion on the surface of an ellipsoid in a flat space remains completely integrable if one adds an elastic force whose line of action always passes through the centre of symmetry of the ellipsoid [1]. The following theorem is a natural analogue of this result of Jacobi.

Theorem 3. The problem of motion on the ellipsoid $E^n \subset S^{n+1}$ under forces with potential

$$V = \sum_{\nu=1}^{n+2} \alpha_\nu / x_\nu^2 \tag{5.3}$$

is completely integrable.

To determine the geometrical meaning of the potential (5.3), let us consider the motion of a point on S^{n+1} under a conservative force whose potential depends on the distance to some centre (a point on S^{n+1}); this is an analogue of central motion in plane Euclidean space. As the distance one can take the angular coordinate ϑ on a great circle, measured from the centre. The following generalized Bertrand problem was solved in [11]: determine all potentials for which almost all orbits are closed. It turns out that this problem (just as in a space of zero curvature) has just two solutions

$$V = \alpha \operatorname{ctg} \vartheta, \quad V = \frac{\beta}{2} \operatorname{tg}^2 \vartheta; \quad \alpha, \beta = \text{const} \tag{5.4}$$

The first of these is an analogue of the Newton potential: it satisfies the Laplace–Beltrami equation on S^3 [3]. The second solution is an analogue of the potential of an elastic spring.

It can be shown that if the centres of elastic attraction or repulsion are placed at the points of S^{n+1} with coordinates

$$(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)$$

then, apart from an unimportant additive constant, the potential of the field of force will be of the form (5.3). One of these points is the centre of the ellipsoid E^n.

The proof of Theorem 3 uses spheroconical coordinates. The separation of variables follows the scheme described in Section 2.

Theorem 3 yields certain new integrable billiards problems. Consider the motion of a particle on the sphere S^{n+1} inside (our outside) the ellipsoid E^n under forces with potential (5.3), on the assumption that collisions at the boundary of E^n are absolutely elastic. It can be shown that this dynamical system with $n + 1$ degrees of freedom is completely integrable: it has $n + 1$ independent commuting integrals which are quadratic functions of the velocities.

If $n = 1$, one can speak of the foci of the ellipse E^1. Placing at these foci gravitational centres whose potentials are defined by the first formula of (5.4), we again obtain integrable billiards inside (outside) E^1 on a two-dimensional sphere. The integrability of elastic billiards inside $E^1 \subset S^2$, without external forces, was established in [12]. Many-dimensional extensions were considered in [10].

In conclusion, we note that similar results are true in spaces of constant negative curvature.

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